CEE 491/591 — Groundwater Hydrology
Assignment 2: Fields, Gradients and PDEs in a Spreadsheet

Arash Massoudieh

Module 2

Submission Instructions

All answers must be submitted as a single **Excel file**.

- Each problem must be solved on a **separate worksheet**.
- **All calculations must be performed using Excel formulas** (no hard-coded numerical results).
- All formulas must be **visible and preserved** in the submitted file.
- Clearly label the given data, intermediate calculations, and final answers, with units for all results.
- Write short explanations in a text cell next to the related calculation.
- Hand derivations (Problems 4a and 5) may be typed into text cells, or photographed and inserted as images in the workbook.
- Upload the `.xlsx` file to **Brightspace** → **Assignments**.

Angles are measured counterclockwise from the $+x$ axis (east), in degrees. Use `ATAN2` (note that Excel's argument order is `ATAN2(x, y)`) and `DEGREES`.

Problem 1: A Head Field on a Grid (25 points)

The hydraulic head in an unconfined sand aquifer near a recharge basin is approximated by

$$h(x, y) = 50 - 0.004\,x + 0.002\,y + 3\exp\left[-\frac{(x-400)^2 + (y-600)^2}{40\,000}\right]$$

with h, x and y in metres, for $0 \le x, y \le 1000$ m.

(a) Build a table of h on a grid with 50 m spacing (x across the columns, y down the rows; 21×21 values) using a single formula copied across the table. Make a contour (or surface) chart of the head field. Where is the head highest, and which way does groundwater flow in general?

(b) Differentiate h by hand and write Excel formulas for the exact $\partial h/\partial x$ and $\partial h/\partial y$. Evaluate them at the point P = (300, 500) m.

(c) Estimate $\partial h/\partial x$ and $\partial h/\partial y$ at P with *central* differences, $\dfrac{h(x+\Delta,y)-h(x-\Delta,y)}{2\Delta}$, for $\Delta =$ 100, 50, 10 and 1 m, and $\partial h/\partial x$ with *forward* differences for the same Δ values. Tabulate the absolute errors against the exact values and plot error against Δ on log–log axes. From the slopes of the lines, show that the forward error is proportional to Δ and the central error to Δ^2.

(d) At P, compute the hydraulic gradient vector, its magnitude, and the direction of groundwater flow. With $K = 12$ m/day and effective porosity 0.30, compute the Darcy flux and the average pore velocity.

(e) A pipeline runs through P toward N30°E (60° from the $+x$ axis). What is the slope of the head surface along the pipeline (m of head per m)? Compare it with $|\nabla h|$ and explain the difference using the angle between the two directions.

Problem 2: The Three-Point Problem (15 points)

Three monitoring wells in a confined aquifer have the following coordinates and heads:

Well	x (m)	y (m)	h (m)
W1	120	80	41.62
W2	930	210	39.05
W3	480	870	41.10

(a) Assume the potentiometric surface is a plane, $h = ax + by + c$. Set up the 3×3 linear system and solve it in Excel with `MMULT` and `MINVERSE`.

(b) Give ∇h, its magnitude, and the direction of groundwater flow.

(c) A fourth well W4 at (600, 450) m reads 40.20 m. Predict the head at W4 from your plane and compare. What could explain the difference?

(d) With $K = 8$ m/day and $n_e = 0.25$, compute the Darcy flux, the pore velocity, and the time for a dissolved tracer to travel 300 m along the flow direction.

Problem 3: Total Differential and Material Derivative (20 points)

Part A: error propagation. The flow through an aquifer cross-section is estimated with Darcy's law, $Q = KA\,\Delta h/L$, using $K = 15$ m/day, $A = 400$ m^2, $\Delta h = 2.0$ m and $L = 250$ m. K is uncertain by $\pm 30\%$, Δh by ± 0.05 m and L by ± 5 m.

(a) Compute Q.

(b) Write the total differential dQ in terms of dK, $d(\Delta h)$ and dL, and evaluate each term. Add their absolute values to get a worst-case uncertainty in Q (in m^3/day and in %).

(c) Which measurement would you improve first to reduce the uncertainty in Q? Why?

Part B: following a water parcel. Along a flow path the concentration of a contaminant is

$$C(x,t) = 100\, e^{-x/150}\,(1 + 0.01\,t)$$

in mg/L, with x in m and t in days. A parcel of water starts at $x = 20$ m at $t = 0$ and moves with the pore velocity $v = 0.8$ m/day.

(d) In a table with a row for each day from 0 to 60, compute the parcel position $x(t)$, the concentration it carries, $C(x(t), t)$, and the concentration recorded by a monitoring well fixed at $x = 20$ m. Plot both against time.

(e) Compute $\partial C/\partial t$, $\partial C/\partial x$ and the material derivative $DC/Dt = \partial C/\partial t + v\,\partial C/\partial x$ along the parcel path. Check your DC/Dt at $t = 30$ days against a central-difference estimate from your column of $C(x(t), t)$.

(f) Explain in two or three sentences why the parcel and the fixed well see such different concentration histories.

Problem 4: Laplace's Equation in Cartesian and Polar Form (25 points)

A well at the origin pumps $Q = 600$ m^3/day from a confined aquifer with transmissivity $T = 150$ m^2/day. Far from the well the head is $h_0 = 30$ m at $R = 500$ m. The steady head is

$$h(x,y) = h_0 + \frac{Q}{2\pi T}\ln\frac{r}{R}, \qquad r = \sqrt{x^2 + y^2}.$$

(a) *By hand:* compute $\partial h/\partial x$, $\partial^2 h/\partial x^2$ and $\partial^2 h/\partial y^2$ in Cartesian coordinates and show that $\nabla^2 h = 0$ for $r > 0$. Then repeat in polar form, $\nabla^2 h = \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dh}{dr}\right)$, and comment on which is easier and why.

(b) Build a table of h on a grid from -100 to 100 m in both x and y with 10 m spacing. Using the five-point formula

$$\nabla^2 h \approx \frac{h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1} - 4h_{i,j}}{\Delta^2},$$

compute the numerical Laplacian at every interior node. Compare it with the size of the individual terms $\partial^2 h/\partial x^2$ at the same nodes. Where is the numerical Laplacian far from zero, and why?

(c) For the points (50, 30), (-70, 40) and (30, -90) m, convert to polar coordinates (r, θ) with `SQRT` and `ATAN2`, and compute h.

(d) For $r = 20$, 50, 100 and 200 m, compute dh/dr exactly and with a central difference, and then the flow crossing a circle of radius r, $Q_r = 2\pi r\, T\, dh/dr$. What do you find, and what physical principle does it express?

Problem 5: Reading PDEs (15 points)

(a) For each equation, state its order, whether it is linear, whether it is steady or transient, and whether it is elliptic, parabolic or hyperbolic: (i) $\dfrac{\partial^2 h}{\partial x^2} + \dfrac{\partial^2 h}{\partial y^2} = 0$; (ii) $\dfrac{S}{T}\dfrac{\partial h}{\partial t} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial h}{\partial r}\right)$; (iii) $\dfrac{\partial C}{\partial t} + v\dfrac{\partial C}{\partial x} = 0$; (iv) $\dfrac{\partial C}{\partial t} + v\dfrac{\partial C}{\partial x} = D\dfrac{\partial^2 C}{\partial x^2} - kC$.

(b) A rectangular confined aquifer, 2 km (east–west) by 1 km, is bounded on the west by a river with a stage of 25 m, on the east by a groundwater divide, and on the north and south by impermeable faults. Sketch the domain and write the boundary condition on each side, naming each type (Dirichlet or Neumann).

(c) Repeat the control-volume argument from Module 2 for a box that also receives recharge N (m/day) through its top. Show that steady flow in a confined aquifer of transmissivity T satisfies $T\nabla^2 h = -N$ (Poisson's equation). Is h now allowed to have a maximum inside the domain? Explain physically.